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Maintaining the required temperature deformations of the viscoelastic disc

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Abstract: *In the process of creating various equipment, machines, mechanisms, devices and other constructions, the technological process of thermal shrink fit is often used. It is reasonable to heat ring discs for fitting them on round axes or in cylindrical holes in order to achieve energy savings by using time-varying power of heat sources. The mathematical model of seating is described using the thermo-viscoselasticity equations, which take into account the heating modes, thermal, elastic and viscous characteristics of the material. The physical relations for a viscoelastic body are presented according to the Maxwell's model. The problem is to heat the disc using heat sources with time-varying heat output so that the displacement on the inner contour becomes equal to the required displacement during heat treatment. The problem of thermal viscoelasticity is solved using the method of small parameter, and the integral equation for finding the law of time-varying specific power of heat sources, which ensures the creation of the required displacement on the inner contour of the disc for a given time, is obtained. The materials presented in this research are aimed at further development of the method of thermo-viscoselasticity in accordance with the solution of two-dimensional problems of optimal control and placement of axisymmetric thermal sources in order to create a certain radial displacement for a certain time with minimal energy consumption.*

Keywords: *thermal viscoelasticity, optimal heating, shrink fit, ring disc, Maxwell model, temperature control.*

1. INTRODUCTION

The technological process of thermal shrink fitting is widely applied in mechanical engineering, transport systems, power equipment, and precision assemblies. The method is based on controlled heating of a ring component to increase its inner diameter, allowing assembly with a shaft or cylindrical element. After cooling, a reliable interference fit is formed. Despite its simplicity, the process requires strict control of temperature fields and deformation levels to ensure dimensional accuracy and structural integrity.

Modern engineering materials frequently exhibit viscoelastic behaviour, especially under elevated temperatures. In such cases, purely elastic models are insufficient for predicting deformation and stress relaxation. Thermo‑viscoelasticity theory provides a more adequate framework, incorporating time‑dependent material response and thermal effects.

The problem of ensuring a prescribed radial displacement within a given heating time is directly related to energy efficiency and process optimisation. Instead of applying constant heat power, it is reasonable to use time‑varying thermal sources. This approach allows reduction of peak energy loads and overall power consumption while maintaining the required deformation level.

The objective of this study is to develop a mathematical model that determines the required temperature field and corresponding heat source power to achieve a specified radial displacement of the inner contour of a viscoelastic disc within a fixed time interval.

2. ANALYSIS OF AVAILABLE RESEARCH RESULTS.

As modern technology develops, it becomes necessary to use materials with new rheological properties [1]. The study of such materials and the analysis of their use in modern engineering have demonstrated the need to apply the methods of the theory of viscoelasticity [2, 3] and elastic-viscosity-plasticity in calculating the strength of various structures [4–8].

In these papers, methods for solving quasi-static problems of the linear theory of thermo-viscoselasticity are considered. Particular attention is paid to the advantages of the approximation method. Methods for solving problems with inhomogeneous temperature fields are also proposed. The basic relations of the nonlinear theory of thermo-viscous elasticity are provided. Various modifications and simplifications of the theory are considered, and the problems of nonisothermal problems of thermo-viscous elasticity in the general physically and geometrically nonlinear theory are considered.

The problems of controlling the temperature movements of a body arising from the need to stabilise the geometry of the working surfaces of technological, energy and other equipment elements during its operation are of great practical importance [9–14].

The materials presented in this paper are aimed at further development of the method of thermal viscoelasticity in accordance with the solution of two-dimensional problems of optimal control and placement of thermal axisymmetric sources of time-varying power in order to create a certain radial displacement in a certain time with minimal energy consumption.

3. SETTING THE RESEARCH PROBLEM.

The objective of this research is to calculate the necessary temperature fields in order to achieve a given deformation in a given time, taking into account the viscosity of the disc material. It is reasonable to heat ring discs for fitting them on round axes or in cylindrical holes for the purpose of achieving energy savings by using time-varying power of heat sources.

The mathematical model of the seating can be described using the thermo-viscoselasticity equations, which take into account the heating modes, thermal, elastic, and viscous characteristics of the material [2, 8–14].

The equilibrium equation of a circular disc in the polar coordinate system is as follows:

$$\frac{\partial \sigma_{11}}{\partial r} + \frac{\sigma_{11} - \sigma_{22}}{r} = 0 \tag{1}$$

Displacements and deformations are related by the following relations

$$\varepsilon_{11} = \frac{\partial u}{\partial r} \qquad \varepsilon_{22} = \frac{u}{r} \tag{2}$$

In equations (1) and (2), the following notations are introduced:

σ_{ij} are components of the stress tensor,
ε_{ij} are components of the strain tensor,
r is polar coordinate at any point of the disc,
u is radial displacement of the disc

The physical relations for a viscoelastic body are presented according to Maxwell's model, which can be used to describe the material properties of important structural elements in general or in parts

$$\dot{\sigma}_{11} + \tau_n^{-1}\sigma_{11} = 2G\dot{\varepsilon}_{11} + K\tau_n^{-1}(\varepsilon_{11} + \varepsilon_{22}) +$$
$$+(K - \frac{2}{3}G)(\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}) + 3K(\dot{T} + \tau_n^{-1}T) \quad (3)$$
$$\dot{\sigma}_{22} + \tau_n^{-1}\sigma_{22} = 2G\dot{\varepsilon}_{22} + K\tau_n^{-1}(\varepsilon_{11} + \varepsilon_{22}) +$$
$$+(K - \frac{2}{3}G)(\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}) + 3K(\dot{T} + \tau_n^{-1}T)$$

$$K = \frac{E}{3(1-2\mu)} \qquad G = \frac{E}{2(1+\mu)}$$

where
$\dot{\sigma}_{ij}$ are components of the stress rate tensor,
$\dot{\varepsilon}_{ij}$ are components of the strain rate tensor,
T, $\dot{T}$ is temperature and its change rate,
$\tau_n = \frac{\eta_n}{G}$ is a stress relaxation time,
η_n is a shear viscosity coefficient,
K is a volumetric elasticity coefficient,
G is a shear modulus,
E is Young's modulus,
μ is Poisson's ratio.

Heat conduction equation for the case of convective heat transfer on the disc surface

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} - m^2 T - \frac{1}{a}\frac{\partial T}{\partial t} + \frac{1}{\lambda_g}W = 0 \quad (4)$$

$$\lambda_g = c\,\rho\,a \qquad m^2 = \frac{\alpha}{\lambda h} \qquad Bi = \frac{2\alpha h}{\lambda}$$

In this relation the following notations are introduced:
c, ρ is specific heat capacity and density of the disc material,
a is a thermal conductivity coefficient,
α is heat transfer coefficient on the disc surfaces,
$z = \pm h$, λ thermal conductivity coefficient of the disc material,
a is a temperature conductivity coefficient,
Bi is a Bio criterion.

In this case, the temperature is calculated from the temperature of the medium. Note that in equation (4), the characteristics $T, W$are averaged over the thickness of the disc.

The boundary force conditions of the problem for the disc are written in the form

$$\sigma_{11} = 0 \text{ when } r = R_1 \text{ and } r = R_2 \quad (5)$$

The conditions of convective heat transfer at the edges are written as follows

$$\frac{\partial T}{\partial r} - kT = 0 \text{ when } r = R_1$$

$$\frac{\partial T}{\partial r} + kT = 0 \text{ at } r = R_2 \quad (6)$$

$$k = \frac{\alpha}{\lambda},$$

where $R_1$and R_2 are inner and outer disk radii, respectively.

At an initial time point, the temperature of the disc is equal to the ambient temperature of the inner and outer radii of the disc.

$$T = 0 \tag{7}$$

At the final moment of time, the displacement of the inner radius of the disc reaches the specified value

$$u(R_1) = u_{зад} \quad \text{at} \quad t = \tau \tag{8}$$

The task is to heat the disc by means of heat sources with thermal power $W(t)$, so that during the heat treatment the displacement on the inner contour becomes equal to the required one. For this purpose, it is necessary to find the dependence of the displacement on the specific power of the heat sources applied to the disc.

4. ANALYSIS OF THE RESULTS

By solving the heat conduction equation under the boundary (6) and time (7) conditions, we obtain the expression for determining the temperature

$$T = \mathrm{a}\sum_{\mathrm{i=1}}^{\infty} e^{-a\lambda_i^2 t} \int_t^0 f_i\,(t)e^{a\lambda_i^2 t}\,\mathrm{dt}\;[\mathrm{MJ}_0(\nu_i r) + Y_0(\nu_i r)] \tag{9}$$

$$f_i\,(t) = \frac{\frac{W(t)}{\lambda_g}\int_{R_1}^{R_2}[\mathrm{MJ}_0(\nu_i r) + Y_0(\nu_i r)]\;rdr}{\int_{R_1}^{R_2}[\mathrm{MJ}_0(\nu_i r) + Y_0(\nu_i r)]^2\;rdr} \tag{10}$$

where

ν_i are the roots of the transcendental equation.

$$[\nu_i J_1(\nu_i R_1) + k\,\mathrm{J}_0(\nu_i R_1)] * [-\nu_i\;\mathrm{Y}_1(\nu_i R_2) + k\,\mathrm{J}_0(\nu_i R_2)] -$$
$$-[-\nu_i\;\mathrm{J}_1(\nu_i R_2) + k\,\mathrm{J}_0(\nu_i R_2)] * [\nu_i\;\mathrm{Y}_1(\nu_i R_1) + k\,\mathrm{J}_0(\nu_i R_1)] = 0 \tag{11}$$

$$M = -\frac{-Y_1(\nu_i R_2)\,\nu_i + Y_0(\nu_i R_2)\,k}{-J_1(\nu_i R_2)\,\nu_i + J_0(\nu_i R_2)\,k} \tag{12}$$

$J_0(x), Y_0(x), J_1(x), Y_1(x)$ are the Bessel function of a real argument [15].

The problem of thermo-viscoselasticity (1), (2), (3), (5) is solved using the method of small parameter, which is chosen as the value of τ_n^{-1}. It is the inverse of the stress relaxation time.

Omitting the bulky calculations, the solution of the problem is given in the zero approximation

$$u(R_1) = \frac{2\alpha_T a}{\left(1 + \frac{4G}{9K}\right)R_1}\sum_{\mathrm{i=1}}^{\infty} F e^{-a\lambda_i^2 t}\int_0^t f_i e(t)^{a\lambda_i^2 t}\;\mathrm{dt} \tag{13}$$

where

$$F = \frac{[\mathrm{MJ}_0(\nu_i r) + Y_0(\nu_i r)]R_1}{{\nu_i}^2} + \frac{{R_1}^2 I_1(\nu_i)}{\frac{4G}{9K}(R_2^2 - R_1^2)} + \\ + \frac{{R_1}^2 I_1(\nu_i) - {R_2}^2 I_3(\nu_i)}{R_2^2 - R_1^2} \tag{14}$$

$$I_1(\nu_i) = I_2(\nu_i) - I_3(\nu_i) \tag{15}$$

$$I_2(\nu_i) = \frac{R_2}{\nu_i}\,[\mathrm{MJ}_1(\nu_i R_2) + Y_1(\nu_i R_2)] \tag{16}$$

$$I_3(\nu_i) = \frac{R_1}{\nu_i}\,[\mathrm{MJ}_1(\nu_i R_1) + Y_1(\nu_i R_1)] \tag{17}$$

Equation (13) relates the displacement on the inner edge of the disc to the specific power of the sources (10), which depends on the heat treatment time. By introducing (13) into condition (8), an integral equation for finding the law of change in the specific power of heat sources over time, which ensures the creation of the required displacement on the inner contour of the disc for a given time τ, is obtained.

5. CONCLUSIONS

A mathematical model of the ring disc heating process for thermal shrink fit has been developed. The model is based on the equations of thermo-viscoelasticity using the Maxwell rheological model, accounting for convective heat transfer on the disc surfaces and the relaxation of stresses over time.

An analytical solution to the heat conduction problem for heat sources with time-varying power has been obtained. The temperature field was determined using Bessel function expansions, establishing a direct relationship between the energy characteristics of the heat sources and the thermal expansion of the disc.

The small parameter method was successfully applied, using the reciprocal of the stress relaxation time as the parameter. This allowed for the solution of the thermo-viscoelastic problem in the zero approximation and the derivation of an integral equation to determine the optimal law for the specific power of heat sources.

The practical significance of this research lies in the ability to precisely control the radial displacement of the inner contour of the disc. The identified power variation law ensures that the required displacement for the thermal fit is achieved within a specified time, improving the energy efficiency of the manufacturing process and ensuring the structural integrity of mechanical connections.

Future research prospects involve the further development of thermo-viscoelastic methods to solve two-dimensional optimal control problems and the strategic placement of axisymmetric heat sources to minimize energy consumption while maintaining required geometric tolerances.

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