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Investigation of the influence of contact interaction conditions on the stress-strain state of an elastic bilayerbody

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Abstract: *The aim of this work is to investigate the influence of complex contact interaction conditions on the stress-strain state of an elastic bilayer body of finite dimensions. The case of loading the body with a perfectly rigid, smooth, flat indenter is considered. The following variants of contact interaction between the layers are compared: perfect bonding, possible delamination of smooth layers without slip, and slip between layers with friction forces according to the Coulomb-Amontons law with possible delamination. The contact, bonding, and slip planes are not known in advance and are determined as a result of solving the problem. Using the theory of variational inequalities, the problem is formulated as a quasi-variational inequality. The inequalities are associated with a sequence of minimization problems for functionals over the set of admissible displacements. At each step of the sequence, the normal stresses on the contact surface are determined based on the results of the previous iteration. The obtained variational problems are discretized using the finite element method. Quadratic rectangular finite elements are used for the calculations. Characteristic features of the stress-strain state depending on the contact conditions between the layers and the indenter are revealed. The influence of the indenter length and the order of arrangement of layers with different mechanical properties on the stress-strain state is considered. Placing the stiffer layer at the bottom promotes a more uniform stress distribution in the body. The lowest stresses in the body occur in the case of perfect bonding of the layers when the stiffer layer is at the bottom. Increasing the indenter size can eliminate lag and significantly reduce interlayer slip.*

Keywords: *bilayer body; indenter; stress-strain state; finite element method; plane strain; contact conditions; delamination; friction forces.*

1. INTRODUCTION

In modern designs of energy apparatuses for rocket and aviation technology, multilayer structures made of various materials are often used to combine the high structural strength of the bearing layer with the special properties of the coatings (corrosion resistance, heat resistance, wear resistance, etc.). The use of such products requires comprehensive information about the stresses at the layers' boundary line, since the combination of layers with different thermomechanical characteristics during operation can lead to delamination. This, in turn, can lead to a disruption in the functionality of the entire structure.

The bibliography of scientific works on the theory of contact problems includes thousands of titles. Within the framework of the classical theory of elasticity, the results are exhaustively presented in monograph [1]. Monograph [2] is in fact an encyclopedia of contact interaction mechanics problems, with an emphasis on problem formulation and their practical application. It is worth mentioning that the Hertz's classical contact theory, proposed in 1882, is only valid for elastic bodies in the absence of friction forces.

One of the most important directions in the development of contact theory is nonlinear problems with unknown boundaries of the contact area. The theory of variational inequalities, initiated in works [3–5], made it possible to effectively solve contact problems with consideration of possible delamination and friction forces. Further development of variational formulations and numerical methods for solving contact problems for multilayer bodies within the framework of elasticity theory is given in works [6–8].

The current state of research on contact problems using variational inequalities is analyzed in monographs [9–11] and the publications cited therein. A detailed description of the results regarding the existence, uniqueness, regularity, and approximate solutions of variational and quasivariational inequalities for static and quasistatic problems is provided. Research on the formulation of such problems, and the development of numerical methods for their solution for elastic and viscoelastic bodies are also presented in works [12–15] and the publications cited therein.

The object of research in this work is to evaluate the influence of contact conditions between the layers on the stress-strain state of a two-layer body.

2. METHODS

2.1. Problem statement

The problem of determining the stress-strain state of an elastic two-layer body of finite size $2b \times h_2$ (Fig. 1) under the action of an absolutely rigid indenter in a state of plane deformation is considered. The height of the layers is equal to h_1 and $h_2 - h_1$ respectfully.

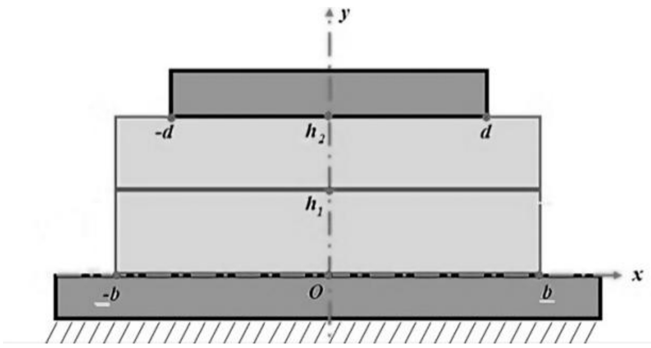


Figure 1. Physical formulation of the problem

We will consider the case of a smooth stamp: contact with the body without taking into account friction forces, with possible separation. For the contact between the layers of the body, we will consider three cases: layers are bound together, possible delamination of smooth layers without friction forces, and the case of layer slippage with friction forces and possible delamination. The body will be considered rigidly bonded to the base. Due to the symmetry of the problem, we will further consider only half of the body with dimensions of b $\times h_2$.

In this problem, plane deformation of the body is characterized by the displacement vector $\vec{u}(x,y) = (u(x,y), v(x,y))$, the components of the strain tensor $\varepsilon_x(x,y), \varepsilon_y(x,y), \varepsilon_{xy}(x,y)$ and the stress tensor $\sigma_x(x,y), \sigma_y(x,y), \sigma_{xy}(x,y)$.

These quantities satisfy the following system of equations

– Equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_x(x,y)}{\partial x} + \dfrac{\partial \sigma_{xy}(x,y)}{\partial y} = 0, \\ \dfrac{\partial \sigma_{yx}(x,y)}{\partial x} + \dfrac{\partial \sigma_y(x,y)}{\partial y} = 0; \end{cases} \quad (1)$$

– Cauchy's relations

$$\begin{cases} \varepsilon_x = \dfrac{\partial u(x,y)}{\partial x}, \varepsilon_y = \dfrac{\partial v(x,y)}{\partial y}, \\ \varepsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial v(x,y)}{\partial x}\right); \end{cases} \quad (2)$$

– Hooke's law

$$\begin{cases} \sigma_x = (\lambda + 2\mu)\varepsilon_x + \lambda\varepsilon_y, \\ \sigma_y = \lambda\varepsilon_x + (\lambda + 2\mu)\varepsilon_y, \\ \sigma_{xy} = 2\mu\varepsilon_{xy}, \end{cases} \quad (3)$$

where $\lambda(x,y)$, $\mu(x,y)$ – Lame coefficients,
with the following boundary conditions

– determined by the problem symmetry

$$u_x(0,y) = 0, \sigma_{xy}(0,y) = 0; \quad (4)$$

– on the right end of the body

$$\sigma_x(b,y) = 0, \sigma_{xy}(b,y) = 0; \quad (5)$$

– on the bottom surface of the body

$$u_x(x,0) = 0, u_y(x,0) = 0; \quad (6)$$

– on the top, free surface of the body

$$\sigma_y(x,h) = 0, \sigma_{xy}(x,h_2) = 0, x \in [d,b] \quad (7)$$

and boundary conditions:

– between indenter and body

$$u_y(x,h_2) \le -\Delta, \sigma_{xy}(x,h_2) = 0, x \in [0,d], \quad (8)$$

where Δ – is a displacement of the stamp, which is determined

– between layers:

- in case of bound layers

$$u_x(x,h_1 - 0) = u_x(h_1 + 0), u_y(x,h_1 - 0) = u_y(x,h_1 + 0); \quad (9)$$

- in case of frictionless sliding with possible delamination

$$\begin{cases} u_y(x,h_1 - 0) \le u_y(x,h_1 + 0), \\ \sigma_y(x,h_1) \le 0, \sigma_{xy}(x,h_1) = 0, \\ \sigma_y(x,h_1)[u_y(x,h_1 - 0) - u_y(x,h_1 + 0)] = 0; \end{cases} \quad (10)$$

- in case of sliding with friction forces according to Coulomb's law with possible delamination

$$\begin{cases} u_y(x,h_1-0) \le u_y(x,h_1+0), \sigma_y(x,h_1) \le 0, \\ |\sigma_{xy}(x,h_1)| < f|\sigma_y(x,h_1)| \Rightarrow u_x(x,h_1-0) = u_x(x,h_1+0), \\ |\sigma_{xy}(x,h_1)| = f|\sigma_y(x,h_1)| \Rightarrow \\ \dfrac{u_x(x,h_1-0)-u_x(x,h_1+0)}{|u_x(x,h_1-0)-u_x(x,h_1+0)|} = -\dfrac{\sigma_{xy}(x,h_1)}{|\sigma_{xy}(x,h_1)|}. \end{cases} \quad (11)$$

Here, f – is the coefficient of friction between the layers.

2.2. Mathematical model of the problem

Using the method of variational inequalities, it was shown in [6] that the problem of determining the stress-strain state (SSS) of contact between multilayer bodies with smooth layers, which are either rigidly bonded or allow for possible sliding and delamination, is equivalent to the problem of minimizing the Lagrangian functional. For the two-layer case it has the following formulation

$$I(v(x,y)) = \sum_{k=1}^{2} \int_{\Omega_k} \left(\frac{1}{2}\lambda_k (\varepsilon_x + \varepsilon_y)^2 + \mu_k (\varepsilon_x^2 + \varepsilon_y^2 + 2\varepsilon_{xy}^2) \right) d\Omega_k \quad (12)$$

on the set of possible displacements $\vec{v}(x,y) = (v_x(x,y), v_y(x,y))$ that satisfy boundary conditions Eq. (4), (6) and contact conditions Eq. (8–10). Here $\Omega_k, \lambda_k, \mu_k\ (k=1,2)$ – is the area occupied by the k-th layer with the Lame parameters of its material respectfully.

In the case of taking friction forces between the body layers into account, the problem will be equivalent to a quasivariational inequality [13]

$$\sum_{k=1}^{2} \int_{\Omega_k} \left[\left(\frac{1}{2}\lambda_k (\xi_x + \xi_y)^2 + \mu_k (\xi_x^2 + \xi_y^2 + 2\xi_{xy}^2) \right) - \left(\frac{1}{2}\lambda_k (\varepsilon_x + \varepsilon_y)^2 + \mu_k (\varepsilon_x^2 + \varepsilon_y^2 + 2\varepsilon_{xy}^2) \right) \right] d\Omega_k +$$

$$+\int_0^b f\,|\sigma_y(x,h_1)| \cdot |v_x(x,h_1-0) - v_x(x,h_1+0)|\,dx \ge 0 \quad (13)$$

on the set V of possible displacements $\vec{v}(x,y) = (v_x(x,y), v_y(x,y))$ that satisfy the displacement boundary conditions Eq. (4), (6) and contact conditions Eq. (8), (11). Possible strains ξ_{ij} are calculated using Cauchy's relations Eq. (2).

It is not possible to associate an extremal problem with quasivariational inequalities (13). However, an iterative process can be used where at each step, the normal stresses σ_y on the contact surface are taken from the previous iteration. The solution of the problem for the case of bonded layers can be chosen as the initial approximation.

Thus, at the n-th step, we obtain a sequence of variational problems of functional minimization

$$I_n(v(x,y)=\sum_{k=1}^{2}\int_{\Omega_k}\left(\frac{1}{2}\lambda_k\left(\varepsilon_x+\varepsilon_y\right)^2+\mu_k\left(\varepsilon_x^2+\varepsilon_y^2+2\varepsilon_{xy}^2\right)\right)d\Omega_k+$$

$$+\int_0^b f\left|\sigma_y^{(n-1)}\left(x,h_1\right)\right|\cdot\left|v_x(x,h_1-0)-v_x(x,h_1+0)\right|dx \qquad (14)$$

on the set of possible displacements $\vec{v}(x,y)=\left(v_x\left(x,y\right),v_y\left(x,y\right)\right)$ that satisfy boundary conditions Eq. (4), (6) and contact conditions Eq. (8), (11). Here n – is the iteration number for the normal stresses on the contact surface.

In functional Eq.(14), the distribution of normal stresses $\sigma_y^{(n-1)}(x,h_1)$ on the layer contact surface is determined from the solution of the problem at the previous step of the iterative process.

2.3. Method of solution

The discretization of the variational minimization problem for functional Eq. (12) (in the absence of friction forces) or the sequence of minimization problems for functionals Eq. (14) (when friction is included) was carried out using the finite element method. Quadratic rectangular elements were used. The accuracy of the numerical solutions was controlled by comparing the results on a sequence of nested finite element meshes. If the results for displacements on two successive meshes improve by less than 5%, the previous mesh size is considered as sufficient.

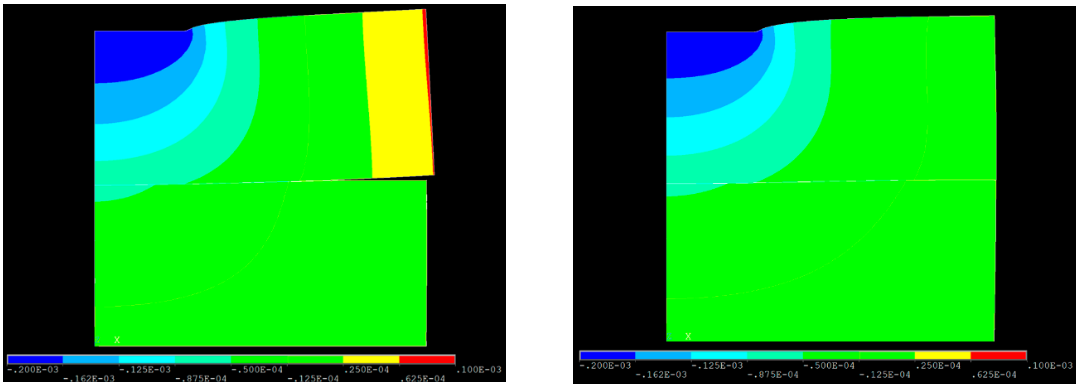
3. RESULTS

A two-layer body with dimensions of *b = 1 m*, h_2 = *1 m*, h_1 = *0,5 m*. was analyzed. The elastic modules of the layers were 200 GPa and 400 GPa. The influence of the layer contact conditions was studied using absolutely rigid indenters of different lengths, *d = 0.3 m* and *d = 0.75 m*. The vertical displacement of the indenter was $\Delta = 0.2$ mm for all considered loading cases. The following contact conditions were examined:

1. Sliding with possible separation.
2. Sliding without separation.
3. Bonding.

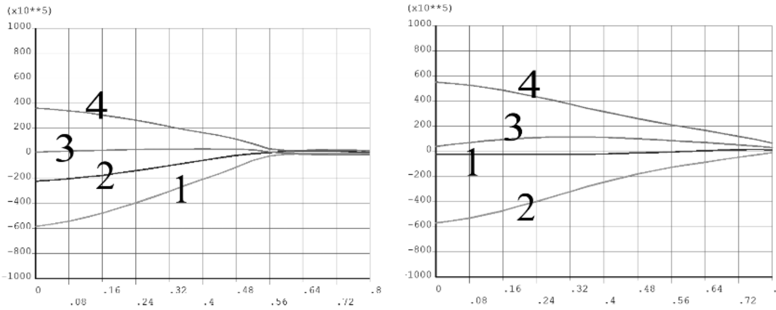
When friction forces were included, the friction coefficient of 0.15 was used between the layers in all cases.

Figure 2 shows the influence of contact conditions on vertical displacements for the first (Fig. 2, a) and third (Fig. 2, b) contact conditions. The end face of thebody, when delamination was possible experienced significant positive vertical displacements of 0.001 m relative to the initial state of the body.



(a) (b)

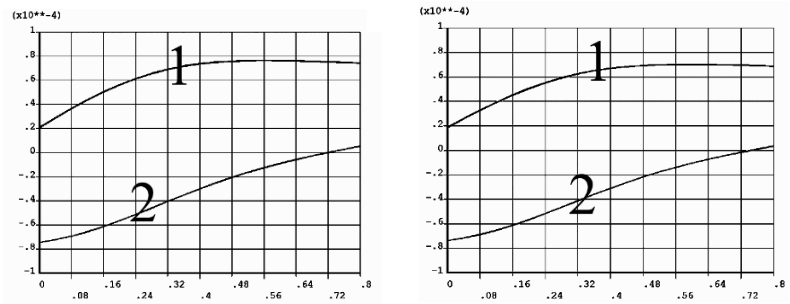
Figure 2. Contour plots of the distribution of vertical displacements: **(a)** For condition of slippage with possible lag; (b) For the adhesion condition.



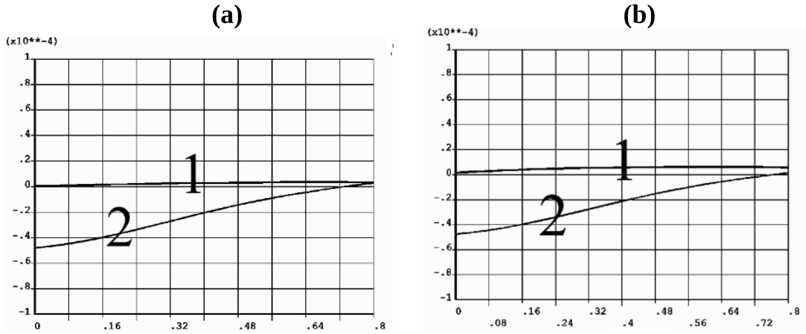
(a) **(b)**

Figure 3. Graphs of stress magnitude along the strip layer above the contact line (a) For condition of slippage with possible lag; (b) For the adhesion condition. σ_x – line 1, σ_y – line 2, σ_{xy} – line 3, σ_i – line 4.

The stress graphs (Fig. 3) above the contact line show the influence of contact conditions 1 and 3.



(a) **(b)**



(c) **(d)**

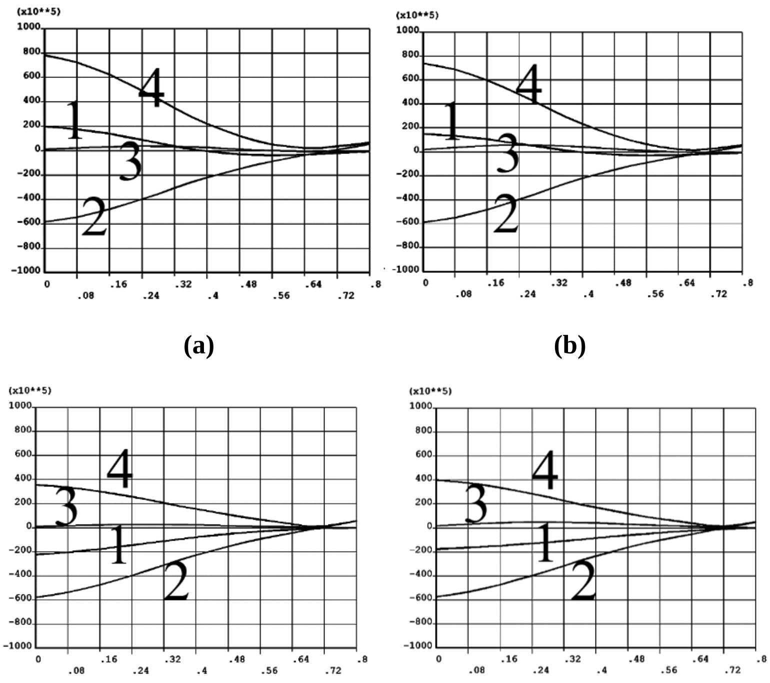
Figure 4. Graphs of the magnitude of vertical and horizontal displacements plotted along the strip layer above and below the contact line: (a) Above the contact line without considering friction forces; (b) Above the contact line considering friction forces; (c) Below the contact line without considering friction forces; (d) Below the contact line considering friction forces. u_x – line 1; u_y – line 2.

Comparing the stress graphs, we see a decrease in intensity at the initial point by quarter. This is due to a reduction in the normal stress σ_x.

Along the boundary of the layers' contact, all stresses in the first option (Fig. 3, a) are close to zero starting from the point x ≈0.6 m due to separation. It is worth noting that the tangential stresses σ_{xy} are almost absent in the case of sliding with possible separation, and a significant increase is observed when the layers are bonded (Fig. 3, b).

The graphs of displacements (Fig. 4) and stresses (Fig. 5) when approaching the layer contact zone for both types of sliding, with and without friction forces, determine the influence of friction forces on the stress-strain state of the body.

From the graphs in Fig. 4 and Fig. 5, it is evident that friction forces do not cause significant changes and have almost no effect on the stresses and displacements in the lower, body that is harder. The area of the upper layer that experiences delamination decreases to a small extent. It is worth noting that in the case of sliding without separation, after the stresses reach zero at the point $x \approx 0{,}7$ m, the normal stress σ_y begins to increase. Due to the inability of the upper layer to detach, positive normal stresses appear on the right end face of the body.



(a) (b) (c) (d)

Figure 5. Graphs of stress magnitude along the strip layer above and below the contact line: (a) For condition of slippage with possible lag, above the contact line without friction; (b) For condition of slippage with possible lag above the contact line with friction; (c) For condition of slippage with possible lag below the contact line without friction; (d) For condition of slippage with possible lag below the contact line with friction; σ_x – line 1, σ_y – line 2, σ_{xy} – line 3, σ_i – line 4.

4. DISCUSSION

The contact conditions between the layers of the body and their physical characteristics introduce significant changes in the body's SSS. This is clearly visible from the graphs showing the distribution of stress intensity within the body (Fig. 2). By increasing the size of the stamp, it is possible to eliminate separation and significantly reduce layer slippage. It's important to note that in all cases, positioning the more rigid layer at the bottom promotes a more uniform stress distribution throughout the entire body and significantly reduces intensity at certain points. The lowest stresses in the body occur when the layers are bonded and the more rigid layer is at the bottom.

Taking into account all the obtained data, including the contribution of friction forces (Fig. 4, Fig. 5), we can conclude that they do not significantly affect the body's SSS, except in some cases where the stress intensity is significantly reduced in specific sections.

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